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Remark about the dependence of the local conservation of the momentum density on the local behaviour of the curl of the superfluid velocity for ^{3}He-A and superfluid ^{4}He

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Abstract. The conservation of the momentum density on the basis of the Pokrovsky–Khalatnikov Hamiltonian technique, suitable for superfluids, is examined. It is demonstrated that the momentum density $j(t, x)$ is conserved when the local behaviour of the curl of the superfluid velocity $v^{(s)}$ is described for ^{3}He-A by the Mermin–Ho formula and for superfluid ^{4}He, in a simply connected region, by the Landau formula. It is conserved also in the case of special relations between j and curl $v^{(s)}$. In the case of the existence of the vortex lines the translational invariance of the system is violated and equations of motion for the momentum density contain extra terms with 'sources' of the momentum.

In this paper we are interested in examination of the conservation of the momentum density for the superfluids ^{3}He-A and He-II. For this purpose it is convenient to use the Hamiltonian technique developed for the hydrodynamics of superfluids (Pokrovsky and Khalatnikov 1976). This technique has been applied to derivation of the hydrodynamic equations for ^{3}He-A (Khalatnikov and Lebedev 1977).

In the monograph by Lamb (1932) the Clebsch transformation of the hydrodynamic equations is presented. The three components of momentum density are expressed in terms of three parameters φ, f, γ

$$\rho v = \rho \nabla \varphi + f \nabla \gamma = j. \tag{1}$$

The parameters f and γ are called the Clebsch variables. Further, after defining the function H, which can play the role of a Hamiltonian, the following equations were derived:

$$\frac{\mathscr{D} f}{\partial t} = -\frac{\partial H}{\partial \gamma} \qquad \frac{\mathscr{D}\gamma}{\partial t} = \frac{\partial H}{\partial f} \qquad \frac{\mathscr{D}}{\partial t} = \frac{\partial}{\partial t} + v \cdot \nabla. \tag{2}$$

They resemble the Hamiltonian equations for canonically conjugate variables (f, γ).

It is worth remembering that the monograph was first published in 1829 (Lamb 1829).

Equations (2) suggest the possibility of developing the Hamiltonian formalism in classical and two-fluid hydrodynamics.

The realisation of this possibility was presented in the papers by Khalatnikov, Lebedev and Pokrovsky (Pokrovsky and Khalatnikov 1976, Khalatnikov and Lebedev 1977). In these papers the expression for the momentum density contains more parameters than are given in (1).

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The hydrodynamics of a normal fluid is described by the conservation laws for the following densities: particles ρ, momentum $\boldsymbol{j}$ and energy E or entropy S.

In the case of superfluids the situation is more complicated and in the hydrodynamic description we should additionally include equations for quasiconserved quantities connected with broken symmetries.

In the case of superfluid ^{4}He the gauge symmetry is broken (additional hydrodynamic equation for the velocity $\boldsymbol{v}^{(s)}$ of the superfluid component).

In the case of superfluid ^{3}He-A, the complications go further. Besides the gauge symmetry, the rotation symmetry in real space is broken, because the direction described by the unit vector $\boldsymbol{l}$ (the intrinsic pair orbital momentum $\boldsymbol{L}^{(i)}$ is expressed in terms of $\boldsymbol{l}$, $\boldsymbol{L}^{(i)} = (\hbar/2m)\rho\boldsymbol{l}$) is locally preferred. Finally, the rotation symmetry in spin space is broken because of the preferred direction described by the unit vector $\boldsymbol{d}$ (additional hydrodynamic equations for $\boldsymbol{v}^{(s)}$, $\boldsymbol{l}$, $\boldsymbol{d}$).

The proposed Hamiltonian technique allowed us to derive, in an elegant way, the hydrodynamic equations for superfluid ^{4}He and ^{3}He-A (Pokrovsky and Khalatnikov 1976, Khalatnikov and Lebedev 1977). Thus the correctness of the proper choice of the canonically conjugated variables was wholly verified. Further, it was shown (Galasiewicz 1987), on the basis of the condition for invariance of the energy density with respect to rotation (Khalatnikov and Lebedev 1977), that the density of the total angular momentum is conserved, although the density of external momentum $\boldsymbol{L}^{(e)}$ and intrinsic angular momentum $\boldsymbol{L}^{(i)}$ are separately not conserved.

Khalatnikov and Lebedev continued investigations of equations of quantum liquids on the basis of canonical equations (Khalatnikov and Lebedev 1978, 1980, Lebedev and Khalatnikov 1977, 1978).

In the present paper, the considerations of the conservation of momentum density have shown that the previously mentioned Hamiltonian technique can give more information about a superfluid system than obtained hitherto. Namely, one can take into consideration, in a natural way, the presence of singularities, i.e. vortex lines.

The Hamiltonian describing ^{3}He-A is of the form (Khalatnikov and Lebedev 1977)

$$H = \int E \, \mathrm{d}^3x \qquad E = \rho\frac{(v^{(s)})^2}{2} + \boldsymbol{p}\boldsymbol{v}^{(s)} + \mathscr{E} \tag{3}$$

where $\mathscr{E}$ is the density of energy in the frame with $\boldsymbol{v}^{(s)} = 0$ and

$$\begin{aligned} \mathrm{d}\mathscr{E} &= \mathrm{d}\mathscr{E}^{(4)} + \frac{\partial\mathscr{E}}{\partial l_i}\,\mathrm{d}l_i + \frac{\partial\mathscr{E}}{\partial\nabla_j l_k}\,\mathrm{d}\nabla_j l_k \\ \mathrm{d}\mathscr{E}^{(4)} &= (v_i^{(n)} - v_i^{(s)})\,\mathrm{d}p_i + T\,\mathrm{d}S + \mu\,\mathrm{d}\rho \end{aligned} \tag{4}$$

where $\boldsymbol{p}$ is defined by (7). The order parameter for ^{3}He-A (without spin variables) is given by a complex vector

$$\boldsymbol{\phi} = \boldsymbol{\phi}_1 + \mathrm{i}\boldsymbol{\phi}_2 \qquad \boldsymbol{\phi}_j\boldsymbol{\phi}_k = \delta_{jk} \qquad j, k = 1, 2. \tag{5}$$

With the help of the order parameter, i.e. $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$, we can define in the A phase some additional quantities like the intrinsic pair orbital momentum $\boldsymbol{l}$ and the superfluid velocity $\boldsymbol{v}^{(s)}$

$$\boldsymbol{l} = \boldsymbol{\phi}_1 \times \boldsymbol{\phi}_2 \qquad \boldsymbol{v}^{(s)} = \nabla\alpha - (\hbar/2m)\boldsymbol{\phi}_2\nabla\boldsymbol{\phi}_1. \tag{6}$$

The equations of motion for $\boldsymbol{l}$ and $\boldsymbol{v}^{(s)}$ should be added to the equations for the conserved densities (ρ, $\boldsymbol{j}$, E).

The momentum density j which is of interest to us is defined by

$$j = p + \rho v^{(s)} \qquad p = S\nabla\beta + f\nabla\gamma. \tag{7}$$

Here f and γ are Klebsch variables. We have in the Hamiltonian (3) three pairs (ρ, α), (S, β), (f, γ) of canonically conjugate variables.

The Hamilton equations of interest to us are

$$\begin{aligned}
&\frac{\partial S}{\partial t} = \frac{\delta H}{\delta\beta} = -\nabla_i(Sv_i^{(n)}) \qquad && \frac{\partial\beta}{\partial t} = -\frac{\delta H}{\delta S} = -T - v_i^{(n)}\nabla_i\beta \\
&\frac{\partial f}{\partial t} = \frac{\delta H}{\delta\gamma} = -\nabla_i(fv_i^{(n)}) && \frac{\partial\gamma}{\partial t} = -\frac{\delta H}{\delta f} = -v_i^{(n)}\nabla_i\gamma \\
&\frac{\partial\rho}{\partial t} = -\nabla_i j_i.
\end{aligned} \tag{8}$$

We are now interested in the equation of motion for j

$$\frac{\partial j}{\partial t} = \frac{\partial p}{\partial t} + v^{(s)}\frac{\partial\rho}{\partial t} + \rho\frac{\partial v^{(s)}}{\partial t}. \tag{9}$$

Equations (8) have exactly the same form for ^{3}He-A and superfluid ^{4}He (Khalatnikov and Lebedev 1977, Pokrovsky and Khalatnikov 1976). On the other hand the equations for $\partial v^{(s)}/\partial t$ are quite different. Therefore we first derive the equation of motion for p defined by (7). We have

$$\begin{aligned}
\frac{\partial p_k}{\partial t} &= \frac{\partial S}{\partial t}\nabla_k\beta + S\nabla_k\frac{\partial\beta}{\partial t} + \frac{\partial f}{\partial t}\nabla_k\gamma + f\nabla_k\frac{\partial\gamma}{\partial t} \\
&= -\frac{\partial}{\partial x_i}\{\delta_{ik}(p_l v_l^{(n)} + TS) + p_k v_i^{(n)}\} + v_l^{(n)}\frac{\partial p_l}{\partial x_k} + T\frac{\partial S}{\partial x_k}.
\end{aligned} \tag{10}$$

We add to the right-hand side of (10) the following identities:

$$\delta_{ik}\frac{\partial}{\partial x_i}(p_l v_l^{(s)}) - v_l^{(s)}\frac{\partial p_l}{\partial x_k} - p_l\frac{\partial v_l^{(s)}}{\partial x_k} = 0 \tag{11}$$

and

$$\begin{aligned}
\frac{\partial p_k}{\partial t} = &-\frac{\partial}{\partial x_i}\{\delta_{ik}[(v_j^{(n)} - v_j^{(s)})p_j + TS] + p_k v_i^{(n)}\} \\
&- p_j\frac{\partial v_j^{(s)}}{\partial x_k} + (v_j^{(n)} - v_j^{(s)})\frac{\partial p_j}{\partial x_k} + T\frac{\partial S}{\partial x_k}.
\end{aligned} \tag{12}$$

As we have mentioned, formula (12) is valid not only for ^{3}He-A but also for superfluid ^{4}He. In the latter case we should put in (4) $l = 0$, $v^{(s)} = \nabla\alpha$ and take in (2) $d\mathscr{E} = d\mathscr{E}^{(4)}$.

We now apply formula (12) to the case of ^{3}He-A. From (2) we have

$$\begin{aligned}
&\frac{\partial\mathscr{E}}{\partial x_k} = \frac{\partial\mathscr{E}^{(4)}}{\partial x_k} + \frac{\partial\mathscr{E}^{(l)}}{\partial x_k} + \frac{\partial}{\partial x_i}\left((\nabla_k l_s)\frac{\partial\mathscr{E}}{\partial\nabla_i l_s}\right) \\
&\frac{\partial\mathscr{E}^{(l)}}{\partial x_k} = \frac{\partial\mathscr{E}}{\partial l_i}\nabla_k l_i - (\nabla_k l_s)\frac{\partial}{\partial x_i}\frac{\partial\mathscr{E}}{\partial\nabla_i l_s}.
\end{aligned} \tag{13}$$

With the help of (13) and (4) the last two terms in (12) can be rewritten as

$$(v_j^{(n)} - v_j^{(s)}) \frac{\partial p_j}{\partial x_k} + T \frac{\partial S}{\partial x_k} = \frac{\partial \mathscr{E}^{(4)}}{\partial x_k} - \mu \frac{\partial \rho}{\partial x_k}$$
$$= \delta_{ik} \frac{\partial \mathscr{E}}{\partial x_i} - \frac{\partial \mathscr{E}^{(l)}}{\partial x_k} - \frac{\partial}{\partial x_i} \left((\nabla_k l_s) \frac{\partial \mathscr{E}}{\partial \nabla_i l_s} + \delta_{ik}(\rho\mu) \right) + \rho \frac{\partial \mu}{\partial x_k}. \tag{14}$$

Equation (12) is now

$$\frac{\partial p_k}{\partial t} = -\frac{\partial}{\partial x_i} \{\delta_{ik}\mathscr{P} + (\nabla_k l_s)\partial\mathscr{E}/\partial\nabla_i l_s + p_k v_i^{(n)}\} - p_l \frac{\partial v_l^{(s)}}{\partial x_k} - \frac{\partial \mathscr{E}^{(l)}}{\partial x_k} - \rho \frac{\partial \mu}{\partial x_k} \tag{15}$$

with

$$\mathscr{P} = TS + \mu\rho + (\boldsymbol{v}^{(n)} - \boldsymbol{v}^{(s)})\boldsymbol{p} - \mathscr{E}. \tag{16}$$

Now, using (15), we are prepared to derive the equation of motion (9) for ^{3}He-A. It follows (Khalatnikov and Lebedev 1977) that

$$\rho \frac{\partial v_k^{(s)}}{\partial t} = -\rho \frac{\partial}{\partial x_k} [\mu + \tfrac{1}{2}(v^{(s)})^2] + \frac{\partial \mathscr{E}^{(l)}}{\partial x_k} - \frac{\hbar}{2m} \varepsilon_{\gamma\alpha\beta} \frac{\partial l_\gamma}{\partial x_l} \frac{\partial l_\alpha}{\partial x_k} l_\beta j_l \tag{17}$$

and

$$v_k^{(s)} \frac{\partial \rho}{\partial t} = -v_k^{(s)} \frac{\partial j_l}{\partial x_l} = -\frac{\partial}{\partial x_l} (j_l v_k^{(s)}) + j_l \frac{\partial v_k^{(s)}}{\partial x_l}. \tag{18}$$

With help of (15), (17) and (18) we finally write equation (9) in the form

$$\frac{\partial j_k}{\partial t} + \frac{\partial}{\partial x_i} \pi_{ki} = j_l \left(\varepsilon_{lka} (\mathrm{curl}\ \boldsymbol{v}^{(s)})_a - \frac{\hbar}{2m} \varepsilon_{\beta\gamma\alpha} l_\beta \frac{\partial l_\gamma}{\partial x_l} \frac{\partial l_\alpha}{\partial x_k} \right) = F_k^{(3)} \tag{19}$$

where the stress tensor π_{ki} is

$$\pi_{ki} = \delta_{ik}\mathscr{P} + j_i v_k^{(s)} + p_k v_i^{(n)} + (\nabla_i l_s) \frac{\partial \mathscr{E}}{\partial \nabla_k l_s} \neq \pi_{ik}. \tag{20}$$

From (19) it follows that for ^{3}He-A the momentum density $\boldsymbol{j}(t, \boldsymbol{x})$ is locally conserved when the local behaviour of curl $\boldsymbol{v}^{(s)}(t, \boldsymbol{x})$ is described by the known Mermin-Ho formula (Mermin and Tin-Lun Ho 1976)

$$\varepsilon_{lka}(\mathrm{curl}\ \boldsymbol{v}^{(s)}(t, \boldsymbol{x}))_a \equiv \frac{\partial v_k^{(s)}(t, \boldsymbol{x})}{\partial x_l} - \frac{\partial v_l^{(s)}(t, \boldsymbol{x})}{\partial x_k}$$
$$= \frac{\hbar}{2m} \varepsilon_{\beta\gamma\alpha} l_\beta(t, \boldsymbol{x}) \frac{\partial l_\gamma(t, \boldsymbol{x})}{\partial x_l} \frac{\partial l_\alpha(t, \boldsymbol{x})}{\partial x_k} \equiv Z_{lk}. \tag{21}$$

For He-II formula (13) gives

$$\frac{\partial \mathscr{E}}{\partial x_k} = \frac{\partial \mathscr{E}^{(4)}}{\partial x_k} \tag{22}$$

where $d\mathscr{E}^{(4)}$ is defined by (4). In addition now in (17) $\partial\mathscr{E}^{(l)}/\partial x_k$ vanishes as well as the last term containing l_β and its derivatives. We have, instead of (17), the Landau equation

$$\rho \frac{\partial v_k^{(s)}}{\partial t} = -\rho \frac{\partial}{\partial x_k} [\mu + \tfrac{1}{2}(v^{(s)})^2]. \tag{23}$$

The equation of motion for the momentum density is now of the form

$$\frac{\partial j_k}{\partial t}+\frac{\partial}{\partial x_i}\pi_{ki}=-\varepsilon_{kla}j_l(\operatorname{curl}\boldsymbol{v}^{(s)})_a=F_k^{(4)}. \tag{24}$$

From (24) it follows that for He-II the momentum density $\boldsymbol{j}(t,\boldsymbol{x})$ is locally conserved with, at $(t,\boldsymbol{x})$,

$$\operatorname{curl}\boldsymbol{v}^{(s)}(t,\boldsymbol{x})=0. \tag{25}$$

This condition is not fulfilled along vortex lines.

It follows from (24) that the momentum density can in addition be conserved in He-II if

$$\boldsymbol{j}\times\operatorname{curl}\boldsymbol{v}^{(s)}=0 \tag{26}$$

but $\operatorname{curl}\boldsymbol{v}^{(s)}\neq 0$. In this case, therefore, $\boldsymbol{j}$ should be parallel to curl $\boldsymbol{v}^{(s)}$ (e.g. should be tangential to a vortex line).

For ^{3}He-A the situation is more complicated. Now if, at some singular points, formula (21) is not fulfilled, the right-hand side of (19) can be equal to zero if

$$-\boldsymbol{j}\times\operatorname{curl}\boldsymbol{v}^{(s)}=\mathscr{Z}\qquad\qquad \mathscr{Z}_k=j_lZ_{lk} \tag{27}$$

where Z_{lk} is defined by (21).

We see that equations (17) and (20) have on the right-hand side the terms

$$\boldsymbol{F}^{(3)}=-(\boldsymbol{j}\times\operatorname{curl}\boldsymbol{v}^{(s)})-\mathscr{Z}\qquad\qquad \boldsymbol{F}^{(4)}=-(\boldsymbol{j}\times\operatorname{curl}\boldsymbol{v}^{(s)}) \tag{28}$$

(where $\mathscr{Z}$ is given by (21) and (27)) which, in the presence of the vortex lines, should in general be different from zero. In that case the momentum density will not be conserved. We must take into consideration the fact that the existence of the vortex lines violates the translation invariance in the system and in this case the momentum density should not be conserved. In the case of non-conservation in (19) and (24) the 'sources' of the momentum ($\boldsymbol{F}^{(3)}$, $\boldsymbol{F}^{(4)}$) should appear. We consider these sources as external forces coming from the vortices and acting on the momentum (i.e. current density $\boldsymbol{j}$). On the other hand, from the point of view of the vortex dynamics, the vortex momentum density should be influenced by forces ($-\boldsymbol{F}^{(3)}$, $-\boldsymbol{F}^{(4)}$) respectively. It means that the total momentum density (current vortices) will be conserved.

In the case of the absence of the vortices ($\boldsymbol{F}^{(3)}=0$) we can treat (19) as a new derivation of the Mermin–Ho formula.

The problem of vortices was presented recently for superfluid ^{4}He (Glaberson and Schwarz 1987) and for ^{3}He-A (Hakonen and Lounasmaa 1987). If $\boldsymbol{l}$ is uniform, $\mathscr{Z}$ vanishes in (27) and, as in He-II, only singular vortices can be created. In the presence of spatially varying $\boldsymbol{l}$ ($\mathscr{Z}\neq 0$) one can have continuous vortex texture (Hakonen and Lounasmaa 1987).

For a classical liquid, the momentum density is defined by (Pokrovsky and Khalatnikov 1976)

$$\boldsymbol{j}=\rho\boldsymbol{v}=\rho\nabla\alpha+\boldsymbol{p}\rightarrow\boldsymbol{v}=\nabla\alpha+\frac{1}{\rho}\boldsymbol{p} \tag{29}$$

and

$$E=\tfrac{1}{2}\rho v^2+\mathscr{E}\qquad\qquad \mathrm{d}\mathscr{E}=T\,\mathrm{d}S+\mu\,\mathrm{d}\rho. \tag{30}$$

In formula (12) one should now put $\boldsymbol{v}^{(s)} = \boldsymbol{v}^{(n)} = \boldsymbol{v}$ and remember that according to (29) $\boldsymbol{v} = \boldsymbol{v}(\rho)$.

With the help of the equation

$$\frac{\partial \alpha}{\partial t} = -\frac{\delta H}{\delta \rho} = -(\mu + \tfrac{1}{2}v^2) + \frac{1}{\rho}\,\boldsymbol{p}\boldsymbol{v} \tag{31}$$

(in the paper of Pokrovsky and Khalatnikov (1976) the last term in (31) is omitted) we can get the proper conservation equation for $\boldsymbol{j}$.

References

Galasiewicz Z M 1987 *Acta Phys. Pol.* A **71** 933

Glaberson W J and Schwarz K W 1987 *Phys. Today* **40** (2) 54

Hakonen P and Lounasmaa O V 1987 *Phys. Today* **40** (2) 70

Khalatnikov I M and Lebedev V V 1977 *Phys. Lett.* **61A** 319

—— 1978 *J. Low Temp. Phys.* **32** 789

—— 1980 *Prog. Theor. Phys. Suppl.* **69** 269

Lamb H 1829 *Treatise on the Mathematical Theory of the Motion of Fluids* (Cambridge: Cambridge University Press)

—— 1932 *Hydrodynamics* (Cambridge: Cambridge University Press) 6th edn

Lebedev V V and Khalatnikov J M 1977 *Zh. Eksp. Teor. Fiz.* **73** 1537 (*Sov. Phys.-JETP* **46** 808)

—— 1978 *Zh. Eksp. Teor. Fiz.* **75** 2312 (*Sov. Phys.-JETP* **48** 1167)

Mermin N D and Tin-Lun Ho 1976 *Phys. Rev. Lett.* **36** 594

Pokrovsky W L and Khalatnikov I M 1976 *Zh. Eksp. Teor. Fiz. Pis. Red.* **23** 653 (*JETP Lett.* **23** 599)